Combinatorial Optimization inspired by Uncertainties

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Uncertainties complicates Optimization

but

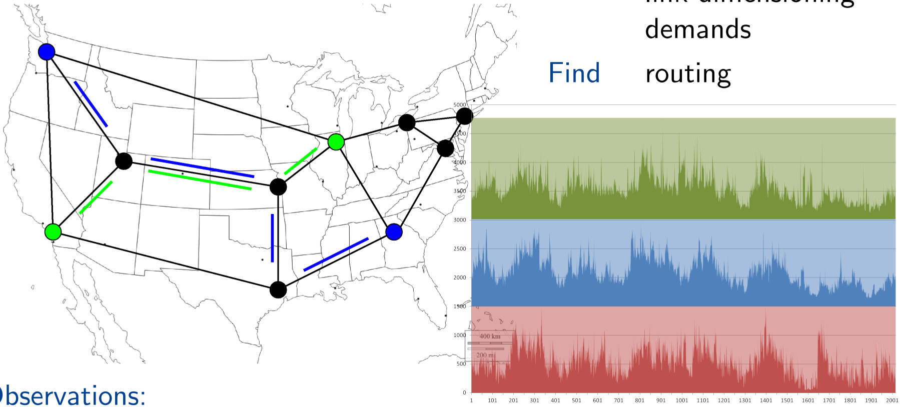
understanding the complexity increase helps (and is fun)

- Case I: developing polyhedral theory further
- Case II: reformulating to known problems
- Case III: determining complexity border

Joint works with Christina Büsing, Timo Gersing, Alexandra Grub, Manuel Kutschka, Wlademar Laube, Nils Spiekermann, Martin Tieves

Motivation: Bandwidth Packing Problem

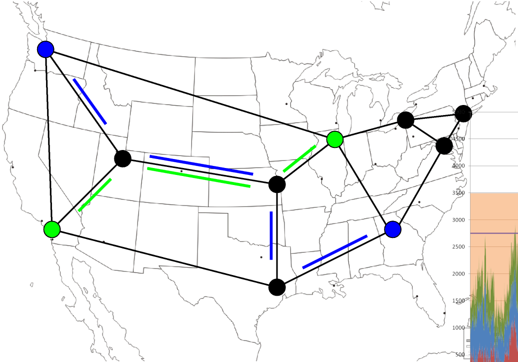
Given	network topology
	link dimensioning
	demands
Find	routing



Observations:

- single path routing
- binary decision on single link → 0-1 Knapsack Problem
- demand values are uncertain

Motivation: Bandwidth Packing Problem



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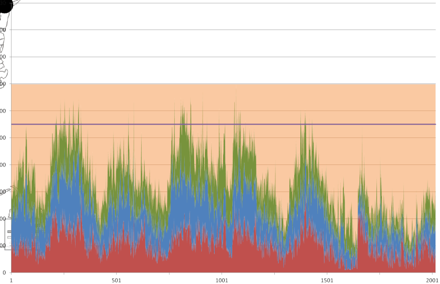

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Robust Optimization according to Ben-Tal and Nemirovski:

Uncertain Linear Program

An Uncertain Linear Optimization problem (ULO) is a collection of linear optimization problems (instances)

$$\left\{\min\{c^T x : Ax \leq b\}\right\}_{(c,A,b)\in\mathcal{U}}$$

where all input data stems from an uncertainty set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$.

Robust Knapsack Problem

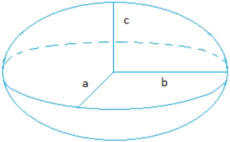
$$\max\left\{c^T x : \{a^T x \leq b, x \in \{0,1\}^n\}_{a\in\mathcal{U}}\right\}$$

How to define $\mathcal{U}$?

How to define the uncertainty set?

- Uncertainty set is an ellipsoid, e.g.,

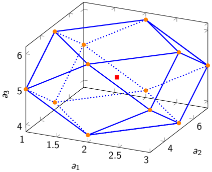
$$\mathcal{U} = \{a \in \mathbb{R}^n : \|a - \bar{a}\| < \kappa\}$$



- Uncertainty set is a polyhedron, e.g.,

$$\mathcal{U} = \{a \in \mathbb{R}^n : D \cdot a \leq d\}$$

with $D \in \mathbb{R}^{k \times n}$, $d \in \mathbb{R}^k$ for some $k \in \mathbb{N}$.
equivalent: set of discrete scenarios (extreme points of polyhedron)
special case: Γ-Robustness;



$$\mathcal{U}(\Gamma) = \left\{ a \in \mathbb{R}^n : a_i = \bar{a}_i + \hat{a}_i \delta_i, \sum_{i=1}^{n} \delta_i \leq \Gamma, \delta \in \{0,1\}^n \right\}$$

Γ-Robust Knapsack polytope:

$$conv\left\{x \in \{0,1\}^{|N|} : \sum_{i \in N} a_i \bar{a}_i x_i + \sum_{i \in S} \hat{a}_i x_i \leq b \ \forall S \subseteq N, |S| \leq \Gamma \right\}$$

Cover inequalities for Knapsack:
Set C with $a(C) > b$:

$$x(C) \leq |C| - 1$$

Extended Cover inequalities:
$E(C) := C \cup \{i : a_i \geq \max_{j \in C} a_j\}$:

$$x(E(C)) \leq |C| - 1$$

How to define covers for Γ-robust knapsack?
$C \subseteq N$ is a $\Gamma-$*robust cover*: $\exists S \subseteq C$ with $|S| \leq \Gamma$ and $\bar{a}(C) + \hat{a}(S) > b$
What about the extension?

Scenario Extension

(C, S) a *cover-pair* if $S \subseteq C$, $|S| \leq \Gamma$, and $\bar{a}(C) + \hat{a}(S) > b$.
Extension for cover-pair (C, S):

$$E(C, S) := C \cup \left\{ i \in N \setminus C : \bar{a}_i \geq \max_{j \in C \setminus S} \bar{a}_j,\ \bar{a}_i + \hat{a}_i \geq \max_{j \in S} (\bar{a}_j + \hat{a}_j) \right\}.$$

Lemma (Büsing, K., Kutschka (2011))

$\sum_{j \in E(C,S)} x_j \leq |C| - 1$ *is a valid inequality for all cover-pairs* (C, S).

Example Scenario Extensions

Scenario Extension

$$E\left(C,S\right) := C \cup \left\{ i \in N : \bar{a}_i \geq \max_{j \in C \setminus S} \bar{a}_j,\ \bar{a}_i + \hat{a}_i \geq \max_{j \in S}(\bar{a}_j + \hat{a}_j) \right\}.$$

$n = 6$ items
$b = 21$ capacity
$\Gamma = 2$ robustness budget

i	1	2	3	4	5	6
$\bar{a}_i$	5	5	3	3	4	5
$\hat{a}_i$	3	3	3	3	4	1

- $C = \{1,2,3,4\}$ robust cover
- $S_1 = \{1,2\}$ and $S_2 = \{3,4\}$ build cover-pairs with $C = \{1,2,3,4\}$
- extensions $E\left(C,S_1\right) = C \cup \{5\}$ and $E\left(C,S_2\right) = C \cup \{6\}$
- but also $\sum_{j \in C \cup \{5,6\}} x_j \leq 3 = |C| - 1$ is valid
- does there exist an extension $E(C) = C \cup \{5,6\}$?

Union of Extensions

$\mathcal{S}(C) := \{S \subseteq C \mid (C, S)$ is a cover-pair$\}$ all cover-pairs with cover C:

$$\mathcal{E}(C) := \bigcup_{S \in \mathcal{S}(C)} E(C, S).$$

Theorem (Gersing, 2017)

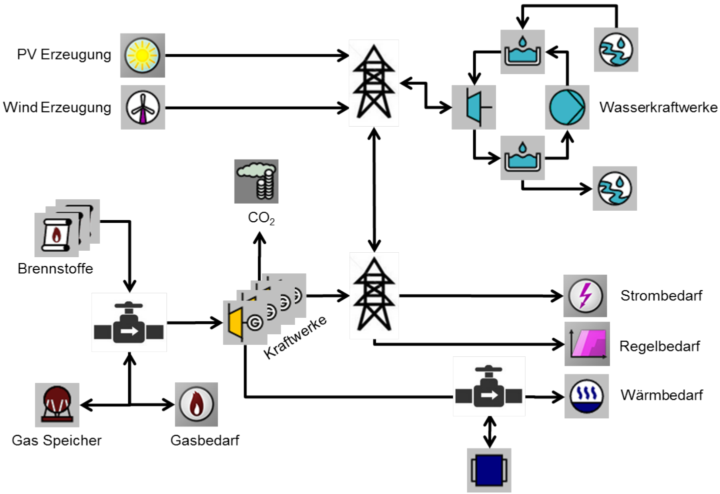
Let $C \subseteq N$ be a $\Gamma-$ robust cover. Then

$$\sum_{j \in \mathcal{E}(C)} x_j \leq |C| - 1$$

is a valid inequality for the Γ-robust knapsack.

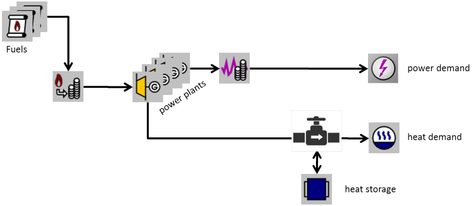
Outline

Energy System schematically



Source: ProCom

Simultaneous production of *heat* and *power* in exchange for *fuel*



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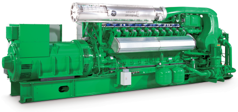


- Fixed ratio ρ between heat and power generation
- Heat can be stored for future use, power cannot be stored
- Heat storage has limited capacity and loss factor

Power has to be bought/sold at day-ahead market!

Lot-Sizing with Storage Deterioration

LS-DET:

$$\min \quad f(q,z) + \sum_{t=1}^{T} h_t u_t \tag{1a}$$

$$\text{s.t.} \quad \alpha u_{t-1} + q_t = u_t + d_t \qquad \forall t \in [T] \tag{1b}$$

$$\underline{U}_t \leq u_t \leq \overline{U}_t \qquad \forall t \in [T] \tag{1c}$$

$$\underline{Q} z_t \leq q_t \leq \overline{Q} z_t \qquad \forall t \in [T] \tag{1d}$$

$$q_t, u_t \geq 0 \qquad \forall t \in [T] \tag{1e}$$

$$z_t \in \{0,1\} \qquad \forall t \in [T] \tag{1f}$$

Lot-Sizing with

- Production limitations
- Storage limitations
- Deterioration of storage
- Concave cost function
- No backlogging

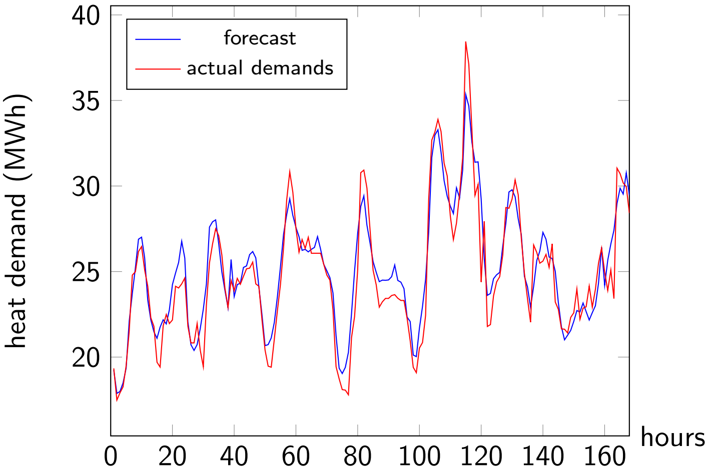
Complexity

- in general: open
- if $\underline{Q} = 0, \overline{Q} = \infty, \alpha = 1$, f linear: LS-DET$\in \mathcal{P}$ (Love, 1973; Atamtürk & Küçükyavuz, 2008)
- if $\underline{U} = 0, \overline{U} = \infty, \alpha = 1$: LS-DET$\in \mathcal{P}$ (Hellion et al., 2012)
- both cases still in $\mathcal{P}$ if $0 < \alpha < 1$ (Schmitz, 2016)

What about uncertain demands?

Forecast & Actual Heat Demands

Heat demands for week 45, 2007



Forecast error of up to 20% (average: 4.1%)

Find solutions that are feasible *with high probability*!

Uncertainty Set: $\mathcal{U}$ of possible demand realizations $(d_t)_{t\in[T]}$

Applying Robust Optimization:

$$\alpha u_{t-1} + q_t = u_t + d_t \tag{1b}$$

Impossible to find (q, z, u) such that (1b)–(1f) are satisfied $\forall d \in \mathcal{U}$

Theorem (folklore)

Every (implicit) equality in $Ax \leq b$ allows for the elimination of a variable involved in the equality.

$\Rightarrow$ In robust optimization, elimination of variable x implies that this variable is moved 2nd stage, i.e., after the uncertain input is known!

RLS-DET:

$$
\begin{aligned}
\min \quad & f(q,z) + \eta && && \text{(2a)}\\
\text{s.t.} \quad & \alpha u_{t-1}(d) + q_t = u_t(d) + d_t && \forall t \in [T], d \in \mathcal{U} && \text{(2b)}\\
& \underline{U} \le u_t(d) \le \overline{U} && \forall t \in [T], d \in \mathcal{U} && \text{(2c)}\\
& \eta \ge \sum_{t \in [T]} h^t u_t(d) && \forall d \in \mathcal{U} && \text{(2d)}\\
& \underline{Q} z_t \le q_t \le \overline{Q} z_t && \forall t \in [T] && \text{(2e)}\\
& q_t, u_t(d) \ge 0 && \forall t \in [T] && \text{(2f)}\\
& z_t \in \{0,1\} && \forall t \in [T] && \text{(2g)}\\
& \eta \ge 0 && && \text{(2h)}
\end{aligned}
$$

- storage $u_t(d)$ per scenario $d \in \mathcal{U}$

Theorem

For an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET can be polynomially reduced (w.r.t. production plans) to an instance of LS-DET with $d = d'$ and $\overline{U} = \overline{U}'$ thus defined:

$$d'_t := \max_{d \in \mathcal{U}} \left\{ d_t - \sum_{i=1}^{t-1} \alpha^{t-i} \left(d'_i - d_i \right) \right\} \qquad \forall t \in [T] \qquad (3a)$$

$$\overline{U}'_t := \overline{U}_t - \max_{d \in \mathcal{U}} \left\{ \sum_{i=1}^{t} \alpha^{t-i} \left(d'_i - d_i \right) \right\} \qquad \forall t \in [T]. \qquad (3b)$$

Corollary

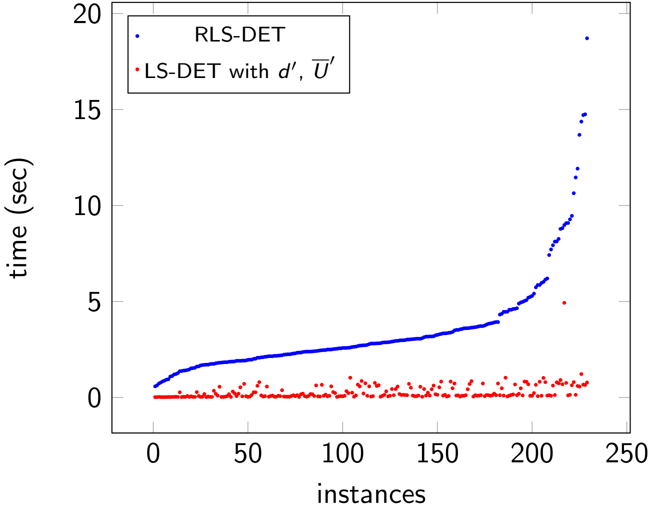
Given an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard) if and only if the corresponding version of LS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard).

Robustness models satisfying precondition:

- polyhedral uncertainty sets, Γ-robustness
- discrete scenarios
- ellipsoidal uncertainty sets

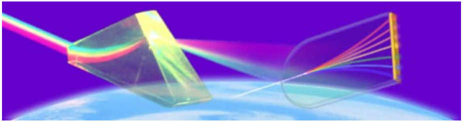
Running times (96h)

Distribution of running times for $|\mathcal{U}| = 50$:

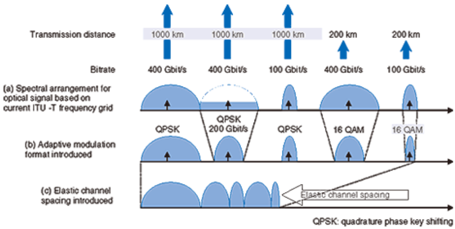


Speed-up factor between 1.82 and 85.67 with average 29.00

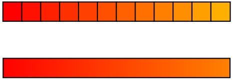
Capacity of optical fibre is huge, but limited!



Idea: More efficient usage of optical channels[1]

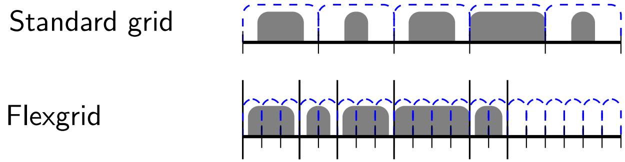


Technology:
Fixed grid
vs.
Flexgrid



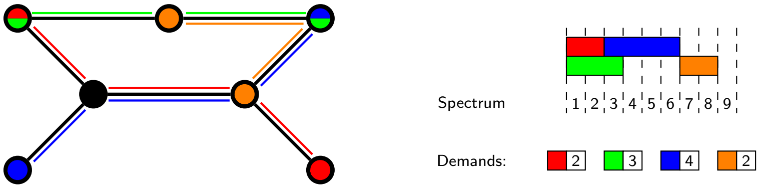
[1] Figure taken from "Innovative Future Optical Transport Network Technologies" by T. Morioka et al., NTT Technical Review, 9 (2011).

Idea: fixed spectrum-block size → flexible block-size



- Spectrum is divided into smaller slots (e.g. 6.25GHz)
- Demands request a custom amount of these slots ('size')
  ⇒ Less spectrum wasted by custom-tailored slot sizes
- "Freedom" is paid for: contiguity of assigned slots required

- In future, demands will be dynamic over time
  ⇒ flexible slot allocation needed
- Question: How to allocate spectrum such that demands can "breath"?



Definition (*Spectrum Allocation Problem* (SA))

Given a simple undirected graph $G = (V, E)$ and a set R of pairs $R_i = (P_i, d_i) \in \mathcal{P} \times \mathbb{N}$, $1 \leq i \leq l$, determine

1. for every R_i an interval $I_i = [a_i, b_i)$ with $a_i \leq b_i \in \mathbb{N}$ und $b_i - a_i = d_i$, such that $\max\{b_i | i = 1, \ldots, l\}$ minimal, where $I_i \cap I_j = \emptyset$ if paths P_i and P_j share an edge in G.

Let $SA(G, R)$ denote the value of an optimal solution.

Lemma (Büsing et al., 2017)

Spectrum Allocation is $\mathcal{NP}$-hard on general networks as well as on star networks

Proof for star networks: wavelength assignment ($d_i = 1$) is $\mathcal{NP}$-hard by a reduction from edge coloring.

Lemma (Büsing et al., 2017)

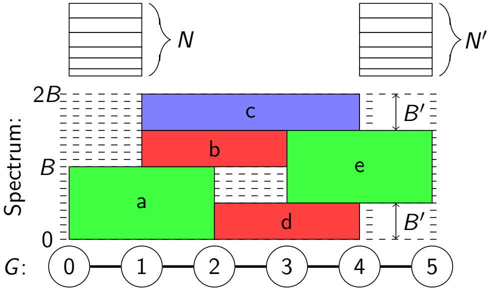
Spectrum Allocation is already $\mathcal{NP}$-hard on path networks and $d_i \in \{1, 2\}$

Proof: Spectrum Allocation on a path is equivalent to Dynamic Storage Allocation, which is known to be $\mathcal{NP}$-hard (GJ, 1979).
Proof for $d_i \in \{c, d\}$ by Ślusarek (1987), corrected by Laube (2017).

Theorem (Büsing et al., 2017)

SA is at least weakly $\mathcal{NP}$-hard, even if G is a path of 5 edges.

Proof: Reduction from PARTITION, $\sum_{i \in N} a_i = B$.



Note: If G is a path of ≤ 3 edges, then SA can be solved in polynomial time.

Robust Spectrum Allocation

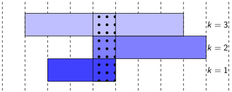
Robust Spectrum Allocation: Given a number of demand scenarios $d^1, \dots, d^K \in \mathbb{Z}_+^{|R|}$, allocate in every scenario the required number of slots such that the total number of slots accross the scenarios is minimized.
$\Rightarrow$ discrete uncertainty set

Applications:

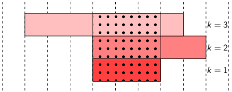
- Prepare for the future: one of the K scenarios will realize, but unknown which one
- Demand will fluctuate between the considered scenarios
- Multi-period Spectrum Allocation with breathing demands

Allocations can breath, but not move (service interruption):
Allocations between scenarios are interwoven!
Any Impact on Optimization?

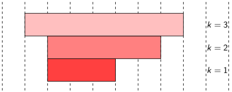
Five (technology) variants:



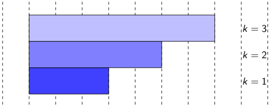
(a) RobSA-A: one joint slot



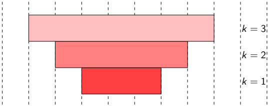
(b) RobSA-B: min. joint slots



(c) RobSA-C: nested



(d) RobSA-D: aligned (left/right)



(e) RobSA-E: overlap in central slot

Lemma

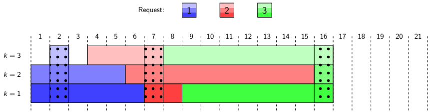
$RobSA_A(G,R) \leq RobSA_B(G,R) \leq RobSA_C(G,R) \leq \min\{RobSA_D(G,R), RobSA_E(G,R)\}$

Robust Spectrum Allocation Strategies

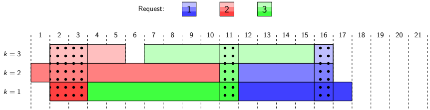
Lemma

There exists instances with
$RobSA_A(G,R) < RobSA_B(G,R) < RobSA_C(G,R) < RobSA_D(G,R)$,
$RobSA_C(G,R) < RobSA_E(G,R)$

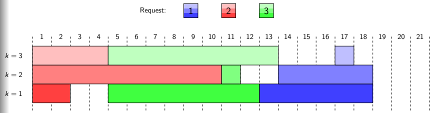
Proof by example:



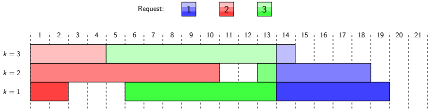
(a) A



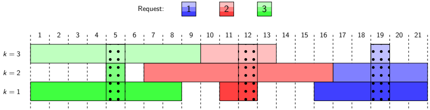
(b) B



(c) C



(d) D



(e) E

Obviously: $RobSA_*(G, R)$ is $\mathcal{NP}$-hard to compute in general networks

What about cases where $SA(G, R)$ is still polynomial solvable?

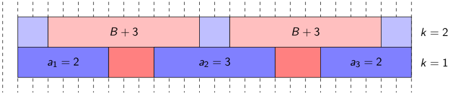
Polynomial solvable cases:

- ??? $|E| = 1$, i.e., single edge case: $SA(G, R) = d(R)$

Theorem (Büsing et al., 2017)

Given a $C \in \mathbb{Z}_+$, the problems whether $RobSA_B(G,R) \leq C$ and $RobSA_C(G,R) \leq C$ are strongly NP-complete, even if $|E| = 1$ and $|K| = 2$.

Reduction from **3-PARTITION**: $3m$ items with size a_i, bound B



Define $5m$ requests with

$$d_r^k := \begin{cases} 2a_r + 2 & \text{if } 1 \leq r \leq 3m,\ k = 1 \\ 2 & \text{if } 1 \leq r \leq 3m,\ k = 2 \\ 3 & \text{if } 3m+1 \leq r \leq 5m,\ k = 1 \\ B+3 & \text{if } 3m+1 \leq r \leq 5m,\ k = 2 \end{cases}$$

Corollary (Büsing et al., 2017)

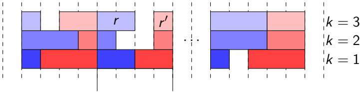
Given a $C \in \mathbb{Z}_+$, the problem whether $RobSA_A(G,R) \leq C$ is strongly NP-complete, even if $|E| = 1$ and $|K| = 2$.

Any good news?

Theorem (Büsing et al., 2017)

$RobSA_D(G, R)$ can be solved in polynomial time on a single link.

Proof:



- Requests are aligned left or right!
- Slots can be saved by combining a left and right request
- Min. weighted perfect matching on complete graph $K_{|R|}$ has to be solved

What about E?

Robust Spectrum Allocation III

Theorem (Büsing et al., 2017)

Let $|K| = 2$ and let d_r^k be odd for all $r \in R$ and $k \in K$. Then, $RobSA_E(G, R)$ on a single link is polynomial-time solvable.

Proof: $RobSA_E$ can be modelled as Gilmore-Gomory-TSP:

NP-complete cases of variants D and E?

Theorem (Büsing et al., 2017)

Given a $C \in \mathbb{Z}_+$, the problem whether $RobSA_D(G, R) \leq C$ is strongly NP-complete, even if $|E| = 2$ and $|K| = 2$.

Reduction from **3-PARTITION**

Theorem (Büsing et al., 2017)

Given a $C \in \mathbb{Z}_+$, the problem whether $RobSA_E(G, R) \leq C$ is strongly NP-complete, even if $|E| = 1$ and $|K| = |R|$ **or** *$|E| = 2$ and $|K| = 2$.*

Reductions from **HAMILTONIAN PATH** and **3-PARTITION**,

Summary

Without uncertainty:

Graph G	Requests R: $d_r = c$	$d_r \in \{c,d\}$	$\lvert P_r\rvert \leq k,\ k \geq 3$	$\lvert P_r\rvert = 3$	$\lvert P_r\rvert \leq 2$
$S_{1,n}$	*str.* $\mathcal{NP}$-c	str. $\mathcal{NP}$-c	str. $\mathcal{NP}$-c	-	str. $\mathcal{NP}$-c
P_n	$\mathcal{P}$	str. $\mathcal{NP}$-c	weak $\mathcal{NP}$-c	weak $\mathcal{NP}$-c	$\mathcal{P}$
$P_n,\ n = 6$	$\mathcal{P}$	open	weak $\mathcal{NP}$-c	$\mathcal{P}$	$\mathcal{P}$
$P_n,\ n = 5$	$\mathcal{P}$	open	open	$\mathcal{P}$	$\mathcal{P}$
$P_n,\ n \leq 4$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$

With uncertainty:

	number of scenarios			
	$\lvert K\rvert = 2$		$\lvert K\rvert = \lvert R\rvert$	general
graph G	A,B,C	D,E	E	D
$\lvert E\rvert = 1$	str. $\mathcal{NP}$-c	$\mathcal{P}$	str. $\mathcal{NP}$-c	$\mathcal{P}$
$\lvert E\rvert \geq 2$	str. $\mathcal{NP}$-c	str. $\mathcal{NP}$-c	str. $\mathcal{NP}$-c	str. $\mathcal{NP}$-c

Concluding Remarks

- Incorporation of Uncertainties in Optimization pays off!
  - ProCom @E-world 2017: BoFiT Optimierung 7.0 – Robust Optimization
- but impacts solution process
- Different ways to model uncertainties yield different results:
  - *Multi-Stage Robustness, Recoverable Robustness, Chance-Constrained Models, Affine Models, etc.*
  - *Evaluation determines feasibility of approach*
- New theory:
  - *Robust* valid inequalities for knapsack, network design, etc.
  - *Robust Lot-Sizing can be solved as deterministic Lot-Sizing*
  - *Complexity border* yields useful insights on robust concepts

Optimization under Uncertainties: just do it!

Combinatorial Optimization inspired by Uncertainties

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